SIMPLE UNIAXIAL AND UNIFORM BIAXIAL DEFORMATION OF NEARLY ISOTROPIC INCOMPRESSIBLE TISSUES

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ABSTRACT A method is developed for analyzing in a unified manner both uniaxial and uniform biaxial strain data obtained from nearly isotropic tissues. The formulation is a direct application of nonlinear elasticity theory pertaining to large deformations. The general relation between Eulerian stress (σ) and extension ratio (λ) in soft isotropic elastic bodies undergoing uniform deformation takes the simple form: $\sigma = ((\lambda^3 - 1)/\lambda)\, f(\lambda)$, where $f(\lambda)$ must be determined for each material. The extension ratio may be either greater than 1.0 (uniaxial elongation), or lie between zero and 1.0 (uniform biaxial extension). Simple analytical functions for $f(\lambda)$ are most readily found for each tissue by plotting all data as $(\lambda^3 - 1)/\lambda\sigma$ vs. λ. Of those tissues investigated in this way (dog pericardium and pleura, and cat mesentery and dura), all but pleura could be adequately described by a parabola: $1/f(\lambda) = 1/k\{[(\lambda_M - \lambda)(\lambda - \lambda_m)]/[\lambda_M - \lambda_m\}$. In these instances, three material constants per tissue (K, λ_M, λ_m) served to predict approximately the stresses attained during both small and large deformations, in strips and sheets alike. It was further found that the uniaxial strain asymptote (λ_M) was linearly related to the biaxial strain asymptote (Λ_M), thus effectively reducing the number of constants by one.

INTRODUCTION

The classical theory of elasticity enables one to calculate all deformations of isotropic continuous solids given just two material constants: either Young's modulus and Poisson's ratio, or the bulk modulus and the shear modulus. These constants can be determined from any single experiment, such as for example, simple elongation or shear, and thereafter applied to all other types of deformation. However, since the classical linear theory is valid only for infinitesimally small strains, it has little application to calculations on nonlinear, easily deformable substances such as elastomers or soft animal tissues. In recent years Rivlin has developed a more general theoretical framework which includes classical theory as a special case (6, 7). The implications of modern finite deformation theory for tissue mechanics are extensive in that, for the first time, it provides some means for systematically analyzing the large-strain properties of complicated materials. It also makes unnecessary the low order approximations found in many solutions based on small-strain concepts.

This paper describes an application of nonlinear finite theory to the analysis of one of the simpler problems in tissue mechanics: uniform deformation of nearly isotropic, almost incompressible tissues. Uniform deformation occurs both during simple uniaxial elongation of a strip (i.e. in the two axes perpendicular to the direction of stretch) and during equal biaxial enlargement of a sheet. However, the results of a two-dimensional large-strain experiment on a tissue sheet cannot in general be predicted from one-dimensional studies on the same tissue. Consequently, both types of experimental data must be obtained on each species of tissue and thereafter possible stress-strain laws may be formulated on an individual basis.

Evidence will be presented that there exist certain types of similarities between the four tissues studied, allowing possible simplification of the description and tentative extrapolation of the results to other nearly isotropic tissues.

MATERIALS AND METHODS

The four tissues selected for this study were cat mesentery, cat dura mater, dog pericardium, and dog pleura. Fresh autopsy specimens were taken from animals previously used for other acute physiological experiments (cardiovascular and neurophysiological). Two methods of study were employed.

A. Narrow strips were dissected having a length of 1.0–4.4 mm and a uniform width of 0.3 to 0.8 mm. Thickness varied from 50 μ for pleura to 300 μ for pericardium. They were then mounted for one-dimensional stretch on the length-tension apparatus previously described in connection with the study of single alveolar walls (2). Wherever forces exceeded the capacity of the original 1.0 g transducer (Sanborn FTA1 Sanborn Co. Waltham, Mass.) a 10.0 g force transducer (Sanborn FTA 10) was substitued.

B. Circular sheets of tissue, having an approximate diameter of 1.5 cm, were stretched biaxially by a ballooning method.[1] Briefly, the sheet is clamped between two plates, the upper of which has an orifice 1.0 cm in diameter. On applying positive pressure beneath the tissue, the sheet balloons through the orifice. Stress is calculated from the pressure, the radius of curvature at the center of the bulge, and the thickness of the sheet. Strain is determined by measuring the relative displacement of microscopic marker granules located near the center of the sheet.

The stress attained at any given deformation depends to some extent on the prior strain rate, on the equilibration period, and on the previous strain history (2, 5). Consequently, these parameters need to be specified. In the one-dimensional experiment, slow cyclic stress-strain loops of increasing amplitude were recorded, until evidence of tearing was seen. For the analysis, one of the records was then chosen whose peak stress was considered safely below the tearing stress. The period of the stretch-release cycles was 30–60 sec. Cycles having longer periods differed negligibly from these. In the two-dimensional strain experiment, the tissues were stretched in a stepwise manner, with measurements being made at each step. The interval was 1–2 min, so that quasi-static values of stress were reached.

The biaxial strain was an average of three strains at directions of about 120° with respect

[1] Hildebrandt, J., H. Fukaya, and C. J. Martin. A method for studying two-dimensional stress-strain relations of tissue sheets. In manuscript.

to each other.[1] The difference between the smallest and largest of these strains was usually about 10% but less than 20%. This indicates the range of measurement error and the effects of slight anisotropy.

Hysteresis was always present, as evidenced by the fact that the stress measured during shortening was somewhat less than that seen during extension. However, it will be assumed that each pair of stress-strain values was sampled in a single-valued equilibrium condition, as has been done for rubber (8) and mesentery (3, 4). Only the stresses seen during lengthening are shown in the data.

THEORY

For an isotropic, incompressible, elastic material deformed isothermally it has been shown (6, 8) that the relations between the three principal stresses (σ_1, σ_2 and σ_3) and the three corresponding extension ratios (λ_1, λ_2 and λ_3) are related to the strain energy per unit volume (W) by

$$\sigma_1 - \sigma_2 = 2(\lambda_1^2 - \lambda_2^2)\left(\frac{\partial W}{\partial I_1} + \lambda_3^2 \frac{\partial W}{\partial I_2}\right) \tag{1}$$

and

$$\sigma_1 - \sigma_3 = 2(\lambda_1^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_2^2 \frac{\partial W}{\partial I_2}\right) \tag{2}$$

where the two strain invariants are given by

$$I_1 = \lambda_1^2 + \lambda_2^2 + 1/\lambda_1^2\lambda_2^2 \tag{3}$$

and

$$I_2 = \lambda_1^2\lambda_2^2 + 1/\lambda_1^2 + 1/\lambda_2^2 . \tag{4}$$

The incompressibility condition yields

$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 = 1. \tag{5}$$

When stress is applied in only one direction (Fig. 1), so that $\sigma_2 = \sigma_3 = 0$ and $\lambda_2 = \lambda_3$, equation 5 becomes $\lambda_1 = 1/\lambda_2^2$, and equations 1 and 2 both reduce to

$$\sigma_1 = 2\left(\lambda_1^2 - \frac{1}{\lambda_1}\right)\left(\frac{\partial W}{\partial I_1} + \frac{1}{\lambda_1}\frac{\partial W}{\partial I_2}\right). \tag{6}$$

When equal stress is applied in two directions simultaneously (Fig. 2 *a*) so that $\sigma_1 = 0$, $\sigma_2 = \sigma_3$, $\lambda_2 = \lambda_3$, and $\lambda_1 = 1/\lambda_2^2$, equations 1 and 3 become

$$-\sigma_2 = 2\left(\lambda_1^2 - \frac{1}{\lambda_1}\right)\left(\frac{\partial W}{\partial I_1} + \frac{1}{\lambda_1}\frac{\partial W}{\partial I_2}\right). \tag{7}$$

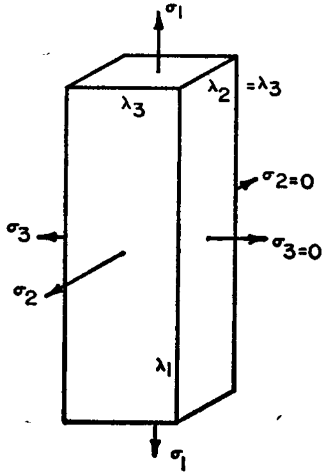

FIGURE 1 Simple one-dimensional elongation (uniaxial deformation).

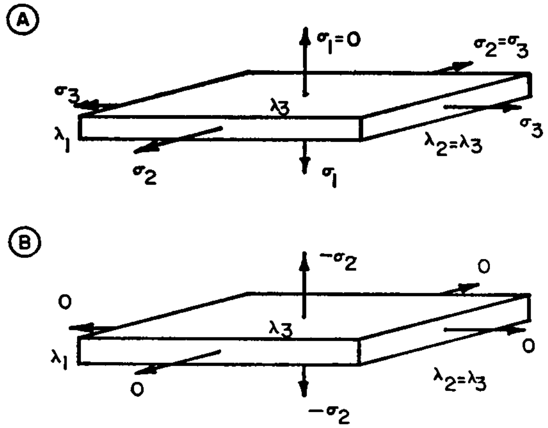

FIGURE 2 Uniform two-dimensional strain (biaxial deformation) of a material isotropic in the plane can be produced in two ways. A. Equal positive stress (σ_2) applied at the edge of the sheet results in a uniform enlargement. B. Negative (compressive) stress ($-\sigma_2$) applied to the face of the sheet produces an identical deformation.

Equations 6 and 7 show that a deformation exactly equivalent to that illustrated in Fig. 2 *a* can be achieved by compressing the sheet with a stress σ_1 equal to $-\sigma_2$ (Fig. 2 *b*). In other words equation 6 may be taken as a general result applicable to simple elongation of a strip when $\lambda > 1.0$, or to uniform extension of a sheet when $\lambda < 1.0$. (With the understanding that $\lambda = \lambda_1$, and $\sigma = \sigma_1$, the subscripts may now be deleted from equation 6). Similarly, the strain invariants for these two particular types of deformation both reduce to

$$I_1 = \lambda^2 + \frac{2}{\lambda} \tag{8}$$

and

$$I_2 = 2\lambda + \frac{1}{\lambda^2}. \qquad (9)$$

To describe completely the elastic behavior of each material, one would like to find the function $W(I_1, I_2)$. However, it can be shown[2] (6, 8) that uniform deformation experiments do not provide sufficient information to determine W completely. A less general formulation of material properties applicable only to uniaxial and uniform biaxial deformation must therefore be developed.

It will be seen from equations 8 and 9 that I_1 and I_2 are functions only of λ. Consequently, $W(I_1, I_2) = W(\lambda)$, and thus $\partial_W/\partial I_1 = \partial_W/\partial\lambda\ \partial\lambda/\partial I_1 = g(\lambda)$, and $\partial_W/\partial I_2 = \partial_W/\partial\lambda \cdot \partial\lambda/\partial I_2 = h(\lambda)$. Therefore, in equation 6

$$2\left(\frac{\partial W}{\partial I_1} + \frac{1}{\lambda}\frac{\partial W}{\partial I_2}\right) = 2\left(g(\lambda) + \frac{1}{\lambda}h(\lambda)\right) = f(\lambda). \qquad (10)$$

Using equation 10, equation 6 may be written as

$$\sigma = \left(\lambda^2 - \frac{1}{\lambda}\right) f(\lambda). \qquad (11)$$

Thus, the function $f(\lambda)$ should incorporate all the characteristics of the material available from our data.

By rearranging equation 11 we have

$$f(\lambda) = \frac{\sigma\lambda}{\lambda^3 - 1}. \qquad (12)$$

Since both σ and λ are measured experimentally, the quantity $\sigma\lambda/(\lambda^3 - 1)$ may be plotted vs. λ, giving $f(\lambda)$ directly. Preliminary work has shown, however, that $\sigma(\lambda)$ has a hyperbolic character[2] so that the reciprocal plot, $(\lambda^3 - 1)/\sigma\lambda$ vs. λ, has some advantages.

One should emphasize that the quantities obtained from deformation of the sheet are σ_2 $(=\sigma_3)$ and λ_2 $(= \lambda_3)$. These can however, be graphed in terms of σ and λ after carrying out the conversion $\sigma = -\sigma_2$ and $\lambda = 1/\lambda_2^2$.

It has been suggested[2] on the basis of attempted curve-fitting, that an equation relating the stress and strain of mesentery and pericardium could take the form

$$\sigma = k\left(\lambda^2 - \frac{1}{\lambda}\right)\left(\frac{1}{(\lambda_M - \lambda)^p} + \frac{1}{(\lambda - \lambda_m)^q}\right) \qquad (13)$$

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[2] Hildebrandt, J., H. Fukaya, and C. J. Martin. Completing the length-tension curve of tissue. In manuscript.

where λ_M and λ_m were maximum and minimum asymptotes, respectively, and k, p, and q were other constants. In the simplest case $p = q = 0$, and equation 13 reduces to one result derived for rubber (8).

Where $p = q = 1$, one obtains from equations 12 and 13:

$$\frac{1}{f(\lambda)} = \frac{\lambda^3 - 1}{\lambda\sigma} = \frac{(\lambda - \lambda_m)(\lambda - \lambda_M)}{k(\lambda_m - \lambda_M)}. \qquad (14)$$

This is a parabola opening downwards, having roots λ_m and λ_M, with a maximum of $(\lambda_M - \lambda_m)/4k$ at $\lambda = (\lambda_m + \lambda_M)/2$. The three constants may therefore be quickly found from the two intercepts on the abscissa and the peak of the parabola.

As a further possibility, letting $p = q = 2$, we have

$$\frac{1}{f(\lambda)} = \frac{(\lambda_M - \lambda)^2(\lambda - \lambda_m)^2}{k[(\lambda_M - \lambda)^2 + (\lambda - \lambda_m)^2]}. \qquad (15)$$

Equation 15 also has roots at λ_m and λ_M, with a maximum at $(\lambda_m + \lambda_M)/2$. Fig. 3

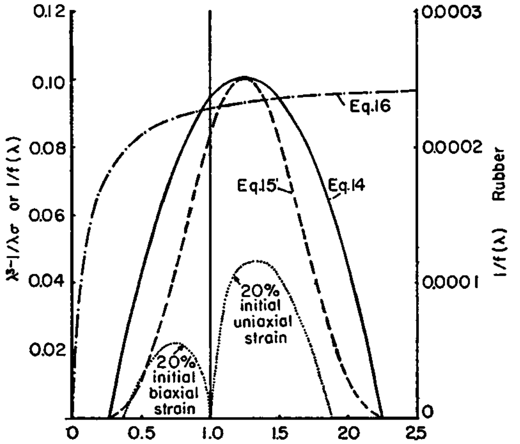

FIGURE 3 Comparison of functions, $1/f(\lambda)$. Equation 14 is calculated for the case

$$\frac{\lambda^3 - 1}{\lambda\sigma} = \frac{1}{5}\left[\frac{(2.25 - \lambda)(\lambda - 0.25)}{(2.25 - \lambda) - (\lambda - 0.25)}\right],$$

and equation 15 for

$$\frac{\lambda^3 - 1}{\lambda\sigma} = \frac{1}{5}\left[\frac{(2.25 - \lambda)^2(\lambda - 0.25)^2}{(2.25 - \lambda)^2 - (\lambda - 0.25)^2}\right].$$

The dotted line illustrates the effects of an error in the resting length amounting to 20%, calculated for the material whose true properties are given by the solid line. Equation 16, which has been applied successfully to rubber, was computed according to $(\lambda^3 - 1)/\lambda\sigma = \frac{1}{400}(\lambda/(10\lambda + 1))$. A separate scale for rubber is shown on the right.

illustrates equations 14 and 15. The characteristic effect of increasing p and q above 1.0 is apparent, thus making it possible to estimate the exponent approximately by visual inspection.

As Fung (3, 4) has pointed out, tissues do appear to have a certain resting length (l_0), but since the stress approaches zero almost asymptotically near l_0, its precise determination is problematic. In the very soft tissues, such as mesentery, errors of 10–20% in l_0 may be readily incurred due to undetected initial strains. Identifying this type of error from the ordinary plot of σ vs. λ is difficult. However, by plotting $1/f(\lambda)$ vs. λ, an error in l_0 shows up in a characteristic fashion. The dotted lines in Fig. 3 show the graphs to be expected if the material represented by the solid line (parabola) is given an initial deformation of 20%, and this deformed dimension is mistakenly taken as l_0. On the basis of the illustration one might regard as suspect any data which shows a sharp curvature toward the abscissa near $\lambda = 1.0$.

RESULTS

All the data obtained from the four types of tissue are plotted in terms of $1/f(\lambda)$ in Fig. 4. Uniaxial deformation is shown on the right half of each graph ($\lambda > 1.0$),

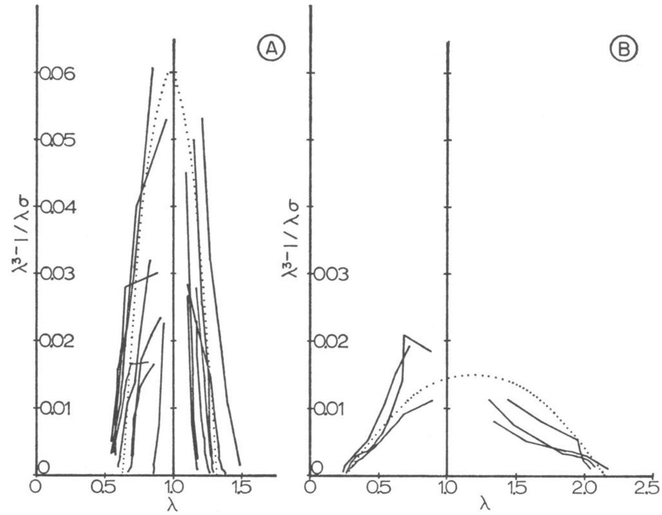

FIGURE 4 Data from four species of tissue sheets plotted in the manner suggested by Fig. 3. Points to the right of $\lambda = 1.0$ represent uniaxial stretch, and points to the left of $\lambda = 1.0$ biaxial stretch. The dotted line in each instance represents a possible parabolic fit using equation 14, with the following constants:

A. Dog pericardium	$K = 2.9$	$\lambda_m = 0.62$	$\lambda_M = 1.32$
B. Dog pleura	$K = 32$	$\lambda_m = 0.25$	$\lambda_M = 2.15$
C. Cat mesentery	$K = 4.2$	$\lambda_m = 0.23$	$\lambda_M = 2.23$
D. Cat dura	$K = 0.7$	$\lambda_m = 0.82$	$\lambda_M = 1.12.$

Data from pleura and dura are probably not conclusive, since sample preparation by peeling is difficult. Furthermore, pleura appears to require a function more complex than a parabola.

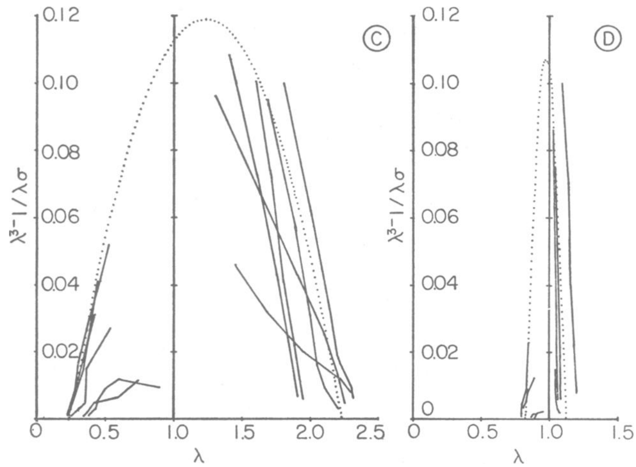

FIGURE 4 C,D

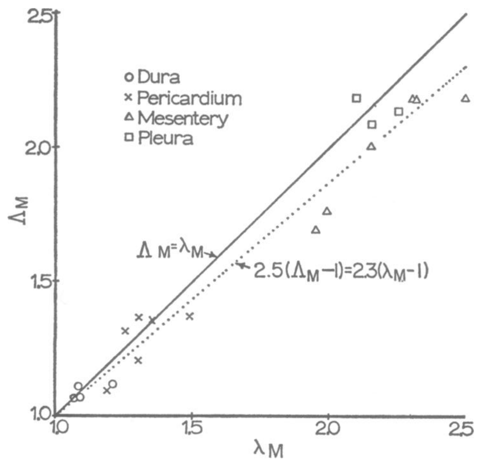

FIGURE 5 Relation between asymptotes, obtained by extrapolation to the abscissa of all curves shown in Fig. 4. The lower asymptote (λ_m) representing greatest thinning of the sheet, is converted to maximal biaxial extension (Λ_M) by using the incompressibility condition (equation 5). The upper asymptote (λ_M) represents the greatest extrapolated extension of the strip. The best linear fit passing through (1, 1) is shown as the dotted line. The latter relation was used to determine λ_m from λ_M when calculating the theoretical curves shown in Fig. 4.

while the biaxial strain is on the left ($\lambda < 1.0$). On each graph a parabola is shown as a dotted line. A number of instances may be noted where an error in l_0 of the type illustrated in Fig. 3 may have appeared. When stresses are very low, reliable measurements cannot be made and thus a gap in the data appears in the neighborhood of $\lambda = 1.0$.

Each individual set of data points was extrapolated downwards to the abscissa. The intercepts yield the two values λ_m and λ_M required by equation 13. The maximum biaxial extension of the sheet Λ_M, may be computed from the greatest thinning of the sheet (λ_m) with the aid of equation 5, $\Lambda_M = 1/\sqrt{\lambda_m}$.

Fig. 5 compares the uniaxial extension asymptotes (λ_M) with the biaxial asymptotes (Λ_M). It appears from this graph that the greatest extension attained uniaxially is only slightly greater than that which can occur if the tissue is stretched in two directions simultaneously.

DISCUSSION

Isotropy

The particular tissues used in this study were selected because fairly homogeneous samples of uniform thickness could be prepared, thus avoiding as much as possible the difficulties of anisotropy and nonuniformity. No systematic anisotropy was, in fact, found in the plane of these tissue sheets. Theoretically, for equation 6 to hold, the isotropy should also apply through the thickness of the sheet, and here we have no data. In the worst case, if isotropy does not apply at all in the third dimension, equation 11 would have to be regarded as a curve-fitting equation unrelated to the finite deformation theory of isotropic materials, and independent of all assumptions.

Incompressibility

It is generally felt that because of their large water content, tissues should be almost incompressible. Indeed, Carew, et al. (1) recently found volume changes in vascular wall at maximum deformation to be less than 0.1 %. Nevertheless, it is possible that some water might be expressed from soft tissues under large uniaxial or biaxial strain. This type of volume change is more difficult to measure, and no reliable information is available. Again, equation 11 could be regarded as an arbitrary descriptive equation in the event that significant compressibility is present.

Uniform Strain

In the immediate vicinity of the clamps, a tissue strip cannot narrow, and therefore cannot undergo uniform strain. These end-effects are negligible if the ratio of strip length to width is about 10 or more. Our ratios were in most instances closer to 4. Consequently, at the largest strains some degree of nonuniformity must have been present, but no account for this complication has been made in the analysis.

Parabolic Fit

In view of the sample variability and experimental error, it seemed to us that the fit should involve as few constants as possible. A parabola with three constants is probably the simplest algebraic expression of $1/f(\lambda)$ which can match the data as closely as seen in Fig. 4. The economy of arbitrary constants is, in fact, unexpected when one considers that equation 11 describes two different types of large deformation and is applicable to several types of tissue. Only pleura would seem to require an expression more complicated than a parabola. At the very high stresses there is some tendency for the data points to spread from the parabola in the manner predicted by equation 15. This would suggest that p and q in equation 13 would both slightly exceed 1.0.

Comparison with Elastomers

Comparison with rubber is often helpful as a reference material. Over most of its range of deformation, rubber can be adequately described as a "Mooney material" (8) by

$$f(\lambda) = C_1 + \frac{1}{\lambda} C_2 \qquad (16)$$

where C_1 and C_2 are constants in the order of 4000 g/cm^2 and 400 g/cm^2, respectively. The reciprocal of this function is shown in Fig. 3, pointing out clearly the differences between elastomers and fresh whole animal tissue. Nevertheless, some components of whole tissue, such as elastin, may have properties closely akin to that of rubber. Tissues are usually composite materials that are architecturally complex and contain many fibers with varying degrees of slack. Consequently the properties of whole tissue cannot be predicted from knowledge of the properties of isolated components, and vice versa.

Strain Asymptotes

The tissues described have characteristics similar to those one might expect to find in an elastic material in which are embedded slack, randomly oriented, almost inextensible cords having few interconnections. Their orientation appears to be random because of the observed near-isotropy. That they are slack but almost inextensible is seen by the sharp increase in modulus at a certain degree of strain which is characteristic of each tissue. From the fact that the maximum biaxial strain is almost as great as the uniaxial, one must conclude that the limiting fibers act almost independently of each other. There is little doubt that this protective network consists of collagen bundles, and that the low extensibility of tissue at high stress represents the recruitment of this component. However, from a phenomenological analysis it is not possible to speculate reliably about the details of molecular or fiber structure, or architecture.

Similarities between Tissues

As shown in Fig. 5, there appears to be a definite relationship between the asymptotes. The relation holds whether the particular tissue extends only 10% like dura, or 130% like mesentery. Knowing the dependence of one variable on another introduces an important simplification, since it reduces the number of arbitrary constants by one. For example, if $f(\lambda)$ contains three parameters, it becomes sufficient to determine only two independent material constants per tissue.

A further similarity can be seen from the presentation used in Fig. 4. There is approximate left-right symmetry of the data points about the vertical midline between the asymptotic intercepts. We know of no theoretical justification for such symmetry. However, its occurrence is convenient in that symmetry enables one to estimate the left half of the function for each tissue knowing only the right half. It usually happens that a uniaxial extension experiment is technically much easier to perform than a uniform biaxial extension.

Finally, the symmetrical function can often be approximated adequately by a parabola involving three and often only two arbitrary constants. In the latter instance both constants are determinable from uniaxial extension data: λ_M from the limiting strain in simple elongation, and k from the peak of the parabola. Thus, in those special cases where approximate symmetry pertains, one might say that biaxial deformation is indeed calculable from uniaxial data. However, this fact must be established separately by experiment for each tissue.

Problem Areas

Since the theory employed here applies to purely elastic media, one cannot expect agreement with data obtained in nonequilibrium or dynamic conditions. Rivlin's general theory (7) makes provision for viscoelasticity, heredity, and memory in materials, but the transition from his mathematical formulation to experimental application has not yet been made.

A less ambitious objective would be to find $W(I_1, I_2)$. Experimentally, this requires nonuniform deformation in two orthogonal directions, e.g., $\lambda_2 \neq \lambda_3$ in Fig. 2 *a*. Even here a suitable experimental technique has not been devised, and further advances in the detailed analysis of tissue mechanics await acquisition of the necessary data.

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